MIDTERM: ALGEBRA III

Date: **11th September 2018**
The Total points is **110** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity** unless specified otherwise.

(1) (15 points) State true or false.
  (a) Every nonzero ring has a prime ideal.
  (b) Every Euclidean domain is a Principal Ideal Domain.
  (c) Every Unique Factorization Domain is a Principal Ideal Domain.
  (d) A prime element in an integral domain is irreducible.
  (e) Two distinct nonzero prime ideals are co-maximal ideals.

(2) (15 points) Let R be an integral domain. Which of the following are an integral domain? No justification needed.
  (a) $R \times R$
  (b) $R[X]$, the polynomial ring over R.
  (c) $S^{-1}R$ where S is a multiplicative subset of R not containing 0.
  (d) Any ring A such that R is a subring of A.

(3) (10 points) Are the two rings $Q[X]/(X^2-1)$ and $Q[X]/(X^2+1)$ isomorphic? Justify your answer.

(4) (3+12=15 points) Define idempotent elements of a ring. Show that the rings $Q[X]/(X^2 - 1)$ and $Q \times Q$ are isomorphic.

(5) (4+16=20 points) Define prime ideals and maximal ideals. Show that in a ring with finitely many elements every prime ideal is maximal ideal.

(6) (15 points) Let k be a field and $R = k[x]$ be the polynomial ring. Let $S = 1, x, x^2, ...$ be a multiplicative subset. Show that the rings $R[y]/(xy-1)$ and $S^{-1}R$ are isomorphic rings.

(7) (4+16=20 points) Define irreducible element and prime element of an integral domain. Let $\mathbb{C}(x, y)$ denote the fraction field of $\mathbb{C}[x, y]$ and let $R = \mathbb{C}(x, y)[z]$ be the polynomial ring over $\mathbb{C}(x, y)$. Show that the ideal generated by the element $y^2z^5 - y^5z^2 + 10xz - 5xy$ in R is a maximal ideal of R.